Gauge Transformations :-

A gauge transformation can be broadly defined as any formal, systematic transformation of the potentials that leaves the fields invariant.

In magnetostatics we know that

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \rightarrow (1)$$

where $\vec{A}$ is called vector potential.

Now from Maxwell's equ^n, we know that

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E}+\frac{\partial \vec{A}}{\partial t}\right) = 0 \rightarrow (2)$$

$\therefore$ $\vec{E}+\frac{\partial \vec{A}}{\partial t}$ can be written as the gradient of some scalar function like

$$\vec{E}+\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \rightarrow (3)$$

[like in electrostatics: $\vec{\nabla}\times\vec{E} = 0$, $\therefore \vec{E} = -\vec{\nabla}\phi$]

ϕ is known as scalar potential.

[In static case, $\frac{\partial \vec{A}}{\partial t} = 0$; $\therefore \vec{E} = -\vec{\nabla}\phi$]

Now substituting $\vec{H} = \vec{B}/\mu$ & $\vec{E} = \vec{D}/\epsilon$ and equ^n (1) & (3) in Maxwell's equ^n $\vec{\nabla}\times\vec{H} = \vec{J}+\frac{\partial \vec{D}}{\partial t}$.

we get,

$$\frac{1}{\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{J} - \epsilon\frac{\partial}{\partial t}\left(\vec{\nabla}\phi+\frac{\partial \vec{A}}{\partial t}\right)$$

[$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}$]

$$\Rightarrow \nabla^2\vec{A} - \mu\epsilon\frac{\partial^2 A}{\partial t^2} = -\mu\vec{J} + \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial \phi}{\partial t}\right] \rightarrow (4)$$

Using eqn (3) in the Maxwell's eqn $\vec{\nabla}\cdot\vec{D} = \rho$ we get.

$$-\epsilon\left[\vec{\nabla}\cdot\vec{\nabla}\phi + \vec{\nabla}\cdot\frac{\partial A}{\partial t}\right] = \rho$$

$$\Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon \longrightarrow (5)$$

We now note that $\vec{A}$ defined by eqn (1) is not unique because $\vec{B}$ remains unchanged if we replace $\vec{A}$ by ~~$A' = A + \vec{\nabla}\psi$~~

$$A' = A + \vec{\nabla}\psi \longrightarrow (6)$$

where ψ is a scalar function.

To keep $\vec{E}$ unchanged as well, ϕ is to be simultaneously changed to

$$\phi' = \phi - \frac{\partial\psi}{\partial t} \longrightarrow (7)$$

This transformation (6) & (7) is called a gauge transformation and the invarience of the fields under such transformation is called gauge invarience.

The freedom implied by eqn (6) & (7) means that we can always choose a set of potentials A, ϕ such that

$$\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial\phi}{\partial t} = 0 \longrightarrow (8)$$

This choice is referred to as the Lorentz gauge.

Substituting the Lorentz gauge in equ^n (4) & (5) we get the uncoupled wave equ^ns for each $\vec{A}$ & $\vec{\phi}$ as.

$$\nabla^2 A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} = -\mu\vec{J} \quad \rightarrow (9)$$

and
$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} = -\rho/\epsilon \quad \rightarrow (10)$$

Here it also be noted that even the potentials satisfying Lorentz condition are not unique. If $\vec{A}$ & $\vec{\phi}$ satisfy the Lorentz condition (8), A' and ϕ' will also satisfy the same condition namely

$$\vec{\nabla}\cdot\vec{A'} + \mu\epsilon\frac{\partial\phi'}{\partial t} = 0.$$

if the gauge function ψ satisfies the homogeneous scalar wave function

$$\nabla^2\psi - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0$$

All potentials in the restricted class are said to belong to the Lorentz gauge.

Let us now consider another useful gauge transformation ie Coulomb gauge which is used in magnetostatics.

As in magnetostatic, we know that

$$\vec{\nabla}\cdot\vec{A} = 0. \quad \rightarrow (11)$$

Thus, equ^n (4) & (5) becomes,

$$\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \mu\epsilon\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \rightarrow (12)$$

and.
$$\nabla^2\phi = -\rho/\epsilon \quad \longrightarrow (13)$$

In the static case, $\frac{\partial \rho}{\partial t} = 0$, ~~this reduces to the old form $\vec{\nabla}\cdot\vec{A} = 0$~~

$\therefore$ Eqn

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \rightarrow (15) \quad \left[\because c^2 = \frac{1}{\mu\epsilon}\right.$$

Eqn (13) can be easily solved to find ϕ as an electrostatic as

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon}\int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}\, dv'$$

But it is difficult to find $\vec{A}$ by solving eqn (12). The Coulomb gauge is often utilized in cases where there is no charge or current distribution ie $\rho = 0$, $\vec{J} = 0$. In this case we take $\phi = 0$ and $\vec{A}$ satisfies the eqn (15)

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 A}{\partial t^2} = 0$$

In this case, the fields are derivable from a single potential

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

Boundary condition =>

from Chattopadhyay & Rakshit - 495.